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Application of Elzaki Transform for Solving Linear Volterra Integral Equations of First Kind

Sudhanshu Aggarwal[1*], Raman Chauhan [2], Nidhi Sharma [3]
[1]*Assistant Professor, Department of Mathematics, National P.G. College Barhalganj,Gorakhpur-273402, U.P., India*
[2,3]*Assistant Professor, Department of Mathematics, Noida Institute of Engineering & Technology,Greater Noida-201306, U.P., India*
sudhanshu30187@gmail.com , raman28chauhan@gmail.com, 6nidhisharma@gmail.com

Abstract: When the advance problems of biology, chemistry, physics and engineering fields represent mathematically then linear Volterra integral equations of first kind appear. In the present research, we used Elzaki transform for solving linear Volterra integral equations of first kind. To demonstrate the effectiveness of Elzaki transform for solving linear Volterra integral equations of first kind, some applications are given in application section.

Keywords: Linear Volterra integral equation of first kind, Elzaki transform, Convolution theorem, Inverse Elzaki transform.

1. INTRODUCTION

The linear Volterra integral equation of first kind is an integral equation in which unknown function occurs only inside the integral sign and it has the form [1-12]

$f(x) = \int_0^x k(x,t)u(t)dt$ …………… (1)

where $u(x)$ is the unknown function, $k(x,t)$ (kernel of integral equation) and the function $f(x)$ are known real-valued functions.

The Elzaki transform of the function $F(t)$ is defined as [16]:

$$E\{F(t)\} = v\int_0^\infty F(t)e^{-t/v}dt$$

$= T(v), t \geq 0, 0 < k_1 \leq v \leq k_2$

where E is Elzaki transform operator.

If $F(t)$ is piecewise continuous and of exponential order then the Elzaki transform of the function $F(t)$ for $t \geq 0$ exist. Both these conditions are sufficient conditions for the existence of Elzaki transform of the function $F(t)$.

Elzaki et al. [17] defined fundamental properties of Elzaki transform together with applications. HwaJoon Kim [18] gave the time shifting theorem and convolution for Elzaki transform. Elzaki and Ezaki [19] discussed the connections between Laplace & Elzaki transforms. Elzaki and Ezaki [20] used Elzaki transform for solving ordinary differential equation with variable coefficients. The solution of partial differential equations using Elzaki transform was given by Elzaki and Ezaki [21]. Shendkar and Jadhav [22] used Elzaki transform for the solution of differential equations. Aggarwal [23] gave the Elzaki transform of Bessel's functions. Aggarwal et al. [24] applied Mahgoub transform for solving linear Volterra integral equations of first kind. Aggarwal et al. [25] gave the application of Elzaki transform for solving population growth and decay problems. Aggarwal et al. [26] used Kamal transform for solving linear Volterra integral equations of first kind. Applications of Mohand transform for solving linear Volterra integral equations of first kind were given by Kumar et al. [27].

The object of present study is to determine the exact solution of linear Volterra integral equation of first kind using Elzaki transform without large computational work.

2. PROPERTIES OF ELZAKI TRANSFORM:

2.1 Linearity property: If $E\{F(t)\} = H(v)$ and $E\{G(t)\} = I(v)$ then

$E\{aF(t) + bG(t)\} = aE\{F(t)\} + bE\{G(t)\}$

$\Rightarrow E\{aF(t) + bG(t)\} = aH(v) + bI(v),$

where a, b are arbitrary constants.

2.2 Elzaki transform of some elementary functions [16-17, 23]:

Table: 1

S.N.	$F(t)$	$E\{F(t)\} = T(v)$
1.	1	v^2
2.	t	v^3
3.	t^2	$2!\,v^4$
4.	$t^n, n \in N$	$n!\,v^{n+2}$
5.	$t^n, n > -1$	$\Gamma(n+1)v^{n+2}$

6.	e^{at}	$\frac{v^2}{1-av}$
7.	$sinat$	$\frac{av^3}{1+a^2v^2}$
8.	$cosat$	$\frac{v^2}{1+a^2v^2}$
9.	$sinhat$	$\frac{av^3}{1-a^2v^2}$
10.	$coshat$	$\frac{v^2}{1-a^2v^2}$

2.3 Inverse Elzaki transform [23]:

If $E\{F(t)\} = T(v)$ then $F(t)$ is called the inverse Elzaki transform of $T(v)$ and mathematically it can be expressed as

$F(t) = E^{-1}\{T(v)\}$

where E^{-1} is the inverse Elzaki transform operator.

2.4 Inverse Elzaki transform of some elementary functions [23]:

Table: 2

S.N.	$T(v)$	$F(t) = E^{-1}\{T(v)\}$
1.	v^2	1
2.	v^3	t
3.	v^4	$\frac{t^2}{2!}$
4.	$v^{n+2}, n \in N$	$\frac{t^n}{n!}$
5.	$v^{n+2}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{v^2}{1-av}$	e^{at}
7.	$\frac{v^3}{1+a^2v^2}$	$\frac{sinat}{a}$
8.	$\frac{v^2}{1+a^2v^2}$	$cosat$
9.	$\frac{v^3}{1-a^2v^2}$	$\frac{sinhat}{a}$
10.	$\frac{v^2}{1-a^2v^2}$	$coshat$

2.5 Convolution theorem for Elzaki transforms [17-18, 23]:

If $E\{F(t)\} = H(v)$ and $E\{G(t)\} = I(v)$ then $E\{F(t) * G(t)\} = \frac{1}{v} E\{F(t)\} E\{G(t)\}$

$\Rightarrow E\{F(t) * G(t)\} = \frac{1}{v} H(v) I(v)$, where $F(t) * G(t)$ is called the convolution of $F(t)$ and $G(t)$ and it is defined as

$$F(t) * G(t) = F * G = \int_0^t F(x) G(t-x) dx$$
$$= \int_0^t F(t-x) G(x) dx$$

2.6 Elzaki transform of Bessel's function of zero order $J_0(t)$[23]:

$$E\{J_0(t)\} = \frac{v^2}{\sqrt{(1+v^2)}}$$

2.7 Elzaki transform of Bessel's function of order one $J_1(t)$ [23]:

$$E\{J_1(t)\} = v - \frac{v}{\sqrt{(1+v^2)}}$$

3. SOLUTION OF LINEAR VOLTERRA INTEGRAL EQUATIONS OF FIRST KIND USING ELZAKI TRANSFORM

In the present work, we will assume that the kernel $k(x,t)$ of linear Volterra integral equation of first kind which is given by (1) is a difference kernel and it can be expressed by the difference$(x-t)$. Thus (1) can be expressed as

$f(x) = \int_0^x k(x-t) u(t) dt$ …………… (2)

Applying the Elzaki transform to both sides of(2), we have

$E\{f(x)\} = E\{\int_0^x k(x-t) u(t) dt\}$……. (3)

Using convolution theorem of Elzaki transform, we have

$$E\{f(x)\} = \frac{1}{v} E\{k(x)\} E\{u(x)\}$$

$\Rightarrow E\{u(x)\} = \left[\frac{vE\{f(x)\}}{E\{k(x)\}}\right]$ … … … … (4)

Operating inverse Elzaki transform on both sides of(4), we have

$u(x) = E^{-1}\left\{\left[\frac{vE\{f(x)\}}{E\{k(x)\}}\right]\right\}$……….. (5)

which is the required solution of (2).

4. APPLICATIONS

To demonstrate the effectiveness of Elzaki transform for solving linear Volterra integral equations of first kind, some applications are given. In these applications, we consider the linear Volterra integral equations of first kind whose kernels containing exponential function, hyperbolic function, trigonometric function, Bessel function etc.

Application: 4.1 Consider linear Volterra integral equation of first kind whose kernel containing exponential function

$x = \int_0^x e^{2(x-t)} u(t) dt$………… (6)

Applying the Elzaki transform to both sides of(6), we have

$$E\{x\} = E\{\int_0^x e^{2(x-t)}\, u(t)dt\}\ldots\ldots\ldots\ldots (7)$$

Using convolution theorem of Elzaki transform on (7), we have

$$v^3 = \frac{1}{v}E\{e^{2x}\}E\{u(x)\}$$

$$\Rightarrow v^3 = \frac{1}{v}\left[\frac{v^2}{1-2v}\right]E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = v^2 - 2v^3 \ldots\ldots\ldots\ldots (8)$$

Operating inverse Elzaki transform on both sides of(8), we have

$$u(x) = E^{-1}\{v^2 - 2v^3\} = E^{-1}\{v^2\} - 2E^{-1}\{v^3\}$$

$$\Rightarrow u(x) = 1 - 2x \ldots\ldots\ldots\ldots (9)$$

which is the required exact solution of (6).

Application: 4.2 Consider linear Volterra integral equation of first kind whose kernel containing exponential function

$$sinx = \int_0^x e^{3(x-t)}u(t)\, dt\ldots\ldots (10)$$

Applying the Elzaki transform to both sides of(10), we have

$$E\{sinx\} = E\{\int_0^x e^{3(x-t)}u(t)\, dt\}\ldots\ldots\ldots\ldots(11)$$

Using convolution theorem of Elzaki transform on(11), we have

$$\frac{v^3}{1+v^2} = \frac{1}{v}E\{e^{3x}\}E\{u(x)\}$$

$$\Rightarrow \frac{v^3}{1+v^2} = \frac{1}{v}\left[\frac{v^2}{1-3v}\right]E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = \frac{v^2(1-3v)}{1+v^2}$$

$$\Rightarrow E\{u(x)\} = \frac{v^2}{1+v^2} - \frac{3v^3}{1+v^2}\ldots\ldots\ldots (12)$$

Operating inverse Elzaki transform on both sides of(12), we have

$$u(x) = E^{-1}\left\{\frac{v^2}{1+v^2}\right\} - 3E^{-1}\left\{\frac{v^3}{1+v^2}\right\}$$

$$\Rightarrow u(x) = cosx - 3sinx \ldots\ldots\ldots\ldots (13)$$

which is the required exact solution of (10).

Application: 4.3 Consider linear Volterra integral equation of first kind whose kernel containing Bessel function of zero order

$$sinx = 4\int_0^x J_0(x-t)u(t)\, dt\ldots\ldots (14)$$

Applying the Elzaki transform to both sides of(14), we have

$$E\{sinx\} = 4E\{\int_0^x J_0(x-t)u(t)\, dt\}\ldots. (15)$$

Using convolution theorem of Elzaki transform on(15), we have

$$\frac{v^3}{1+v^2} = 4.\frac{1}{v}E\{J_0(x)\}E\{u(x)\}$$

$$\Rightarrow \frac{v^3}{1+v^2} = 4.\frac{1}{v}\left[\frac{v^2}{\sqrt{(1+v^2)}}\right]E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = \frac{1}{4}\left[\frac{v^2}{\sqrt{(1+v^2)}}\right]\ldots\ldots\ldots\ldots (16)$$

Operating inverse Elzaki transform on both sides of(16), we have

$$u(x) = \frac{1}{4}E^{-1}\left\{\frac{v^2}{\sqrt{(1+v^2)}}\right\} = \frac{1}{4}J_0(x)\ldots\ldots(17)$$

which is the required exact solution of (14).

Application: 4.4 Consider linear Volterra integral equation of first kind whose kernel linear in the argument x and t

$$x^2 = \frac{1}{3}\int_0^x (x-t)u(t)\, dt\ldots\ldots (18)$$

Applying the Elzaki transform to both sides of(18), we have

$$E\{x^2\} = \frac{1}{3}E\{\int_0^x (x-t)u(t)\, dt\}\ldots. (19)$$

Using convolution theorem of Elzaki transform on(19), we have

$$2!\, v^4 = \frac{1}{3}.\frac{1}{v}E\{x\}E\{u(x)\}$$

$$\Rightarrow 2!\, v^4 = \frac{1}{3}.\frac{1}{v}[v^3]E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = 6v^2 \ldots\ldots\ldots\ldots (20)$$

Operating inverse Elzaki transform on both sides of(20), we have

$$u(x) = 6E^{-1}\{v^2\} = 6\ldots\ldots\ldots\ldots (21)$$

which is the required exact solution of (18).

Application: 4.5 Consider linear Volterra integral equation of first kind whose kernel containing hyperbolic cosine function

$$sinx = \int_0^x \cosh(x-t)\, u(t)dt\ldots\ldots(22)$$

Applying the Elzaki transform to both sides of (22), we have

$$E\{sinx\} = E\{\int_0^x \cosh(x-t)\, u(t)dt\}\ldots. (23)$$

Using convolution theorem of Elzaki transform on(23), we have

$$\frac{v^3}{1+v^2} = \frac{1}{v}E\{coshx\}E\{u(x)\}$$

$$\Rightarrow \frac{v^3}{1+v^2} = \frac{1}{v}\left[\frac{v^2}{1-v^2}\right]E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = \frac{v^2(1-v^2)}{(1+v^2)} = \frac{2v^2}{1+v^2} - v^2\ldots\ldots (24)$$

Operating inverse Elzaki transform on both sides of(24), we have

$$u(x) = E^{-1}\left\{\frac{2v^2}{1+v^2} - v^2\right\}$$

$$= 2E^{-1}\left\{\frac{v^2}{1+v^2}\right\} - E^{-1}\{v^2\}$$

$$\Rightarrow u(x) = 2cosx - 1\ldots\ldots\ldots\ldots (25)$$

which is the required exact solution of (22).

Application: 4.6 Consider linear Volterra integral equation of first kind whose kernel containing cosine function

$$x = \int_0^x cos(x-t)\, u(t)dt \ldots\ldots (26)$$

Applying the Elzaki transform to both sides of (26), we have

$$E\{x\} = E\left\{\int_0^x cos(x-t)\, u(t)dt\right\} \ldots. (27)$$

Using convolution theorem of Elzaki transform on(27), we have

$$v^3 = \frac{1}{v} E\{cosx\} E\{u(x)\}$$

$$\Rightarrow v^3 = \frac{1}{v}\left[\frac{v^2}{1+v^2}\right] E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = v^2 + v^4 \ldots\ldots\ldots\ldots (28)$$

Operating inverse Elzaki transform on both sides of(28), we have

$$u(x) = E^{-1}\{v^2+v^4\} = E^{-1}\{v^2\} + E^{-1}\{v^4\}$$

$$\Rightarrow u(x) = 1 + \frac{x^2}{2} \ldots\ldots\ldots\ldots (29)$$

which is the required exact solution of (26).

Application: 4.7 Consider linear Volterra integral equation of first kind whose kernel containing Bessel function of zero order

$$J_0(x) - cosx = \int_0^x J_0(x-t) u(t)\, dt (30)$$

Applying the Elzaki transform to both sides of (30), we have

$$E\{J_0(x)\} - E\{cosx\} = E\left\{\int_0^x J_0(x-t)u(t)\, dt\right\} .. (31)$$

Using convolution theorem of Elzaki transform on(31), we have

$$\frac{v^2}{\sqrt{v^2+1}} - \frac{v^2}{1+v^2} = \frac{1}{v} E\{J_0(x)\} E\{u(x)\}$$

$$\Rightarrow \frac{v^2}{\sqrt{v^2+1}} - \frac{v^2}{1+v^2} = \frac{1}{v} . \frac{v^2}{\sqrt{v^2+1}} . E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = v - \frac{v}{\sqrt{(1+v^2)}} \ldots\ldots\ldots\ldots (32)$$

Operating inverse Elzaki transform on both sides of(32), we have

$$u(x) = E^{-1}\left\{v - \frac{v}{\sqrt{(1+v^2)}}\right\} = J_1(x) \ldots\ldots (33)$$

which is the required exact solution of (30).

Application: 4.8 Consider linear Volterra integral equation of first kind whose kernel containing Bessel function of order one

$$cosx - J_0(x) = -\int_0^x J_1(x-t)u(t)\, dt \ldots.. (34)$$

Applying the Elzaki transform to both sides of (34), we have

$$E\{cosx\} - E\{J_0(x)\} = -E\int_0^x J_1(x-t)u(t)\, dt \ldots\ldots\ldots\ldots (35)$$

Using convolution theorem of Elzaki transform on(35), we have

$$\frac{v^2}{1+v^2} - \frac{v^2}{\sqrt{v^2+1}} = -\frac{1}{v} E\{J_1(x)\} E\{u(x)\}$$

$$\Rightarrow \frac{v^2}{1+v^2} - \frac{v^2}{\sqrt{v^2+1}} = -\frac{1}{v}\left[v - \frac{v}{\sqrt{(1+v^2)}}\right] . E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = \frac{v^2}{\sqrt{v^2+1}} \ldots\ldots\ldots\ldots (36)$$

Operating inverse Elzaki transform on both sides of(36), we have

$$u(x) = E^{-1}\left\{\frac{v^2}{\sqrt{v^2+1}}\right\} = J_0(x) \ldots\ldots\ldots\ldots (37)$$

which is the required exact solution of (34).

5. CONCLUSION

In the present paper, we have successfully defined the Elzaki transform for solving linear Volterra integral equations of first kind. The given applications showed that very less computational work and a very little time needed for finding the exact solution of linear Volterra integral equations of first kind. In future, Elzaki transform can be applied for solving the system of linear Volterra integral equations.

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Authors Profile:

Sudhanshu Aggarwal received his M.Sc. degree from M.S. College, Saharanpur in 2007. He has also qualified CSIR NET examination (June-2010, June-2012, June-2013, June-2014 & June-2015) in Mathematical Sciences. He is working as an Assistant Professor in National P.G. College Barhalganj Gorakhpur. He is equipped with an extraordinary caliber and appreciable academic potency. He has around ten years of teaching experience at various engineering colleges affiliated to AKTU. His fields of interest include Integral Transform Methods, Differential and Partial Differential Equations, Integral Equations and Number Theory. He has published many research papers in national and international journals.

Raman Chauhan has completed M.Sc. (Mathematics) from Indian Institute of Technology, Roorkee in 2011.He is working at NIET, Greater Noida since 2013 as a dynamic, successful and responsible faculty member in the department of Mathematics. He has also qualified IIT Jam exam (2009) and Gate exam (2011, 2013, 2014 & 2015).His fields of interest include Optimization, Integral Equations and Abstract Algebra.

Nidhi Sharma received her M.Sc. degree from Banasthali University Rajasthan in 2015. She is working as an Assistant Professor in Noida Institute of Engineering & Technology, Greater Noida since 2015. She has also qualified IIT Jam exam (2013).She has published many research papers in national and international journals. Her fields of interest include Optimization, Fuzzy Logic and Topological Spaces.